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Referring now to (1) we get by adding

$$s_1 = 2R\ (\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C). \qquad (4)$$

Hencs $s_2 = 2R_1[\cos(\tfrac{1}{2})^2(\pi-A)+\cos(\tfrac{1}{2})^2(\pi-B)+\cos(\tfrac{1}{2})^2(\pi-C)]$

. . . .

$$s_{n-1} = 2R_{n-2}[\cos(\tfrac{1}{2})^{n-1}(g''\pi\pm A)+\cos(\tfrac{1}{2})^{n-1}(g''\pi\pm B)+\cos(\tfrac{1}{2})^{n-1}(g''\pi \pm C)] \quad (5')$$

$$s_n = 2R_{n-1}[\cos(\tfrac{1}{2})^n(g'\pi\mp A)+\cos(\tfrac{1}{2})^n(g'\pi\mp B)+\cos(\tfrac{1}{2})^n(g'\pi\mp C)]. \quad (5)$$

It is well known that $R_1 = 2R$

whence $R_2 = 2R_1 = 2^2R$

$R_3 = 2R_2 = 2^3R$

. . . .

$$R_{n-1} = 2R_{n-2} = 2^{n-1}R \qquad (6')$$

$$R_n = 2R_{n-1} = 2^nR. \qquad (6)$$

Substituting in (3) the values of s_{n-1} and $2R_{n-1}$ found in (5′) and (6) we have

$$K_n=2^{2n-1}R^2[\cos(\tfrac{1}{2})^{n-1}(g''\pi\pm A)+\cos(\tfrac{1}{2})^{n-1}(g''\pi\pm B)+\cos(\tfrac{1}{2})^{n-1}(g''\pi\pm C)],$$

the argument for g'' being $n-2$.

Example. Suppose it is required to find the area of the 10th derived triangle, the original triangle being equilateral and its sides unity.

We have $R^2 = a^2\div 4\sin^2A = \frac{1}{3}$; $g'' = 85$ and the upper sign is to be used because n is *even*; hence

$$K_{10} = 2^{19}.\cos(\tfrac{1}{2})^9(85\pi+\tfrac{1}{3}\pi) = 2^{19}.\cos\tfrac{\pi}{6} = 2^{18}\sqrt{3}.$$

Cor. If we refer to the angles of the nth triangle we see that as g is increased the angles differ less and less from one another and hence if g is indefinitely increased the inequality of the angles is indefinitely diminished or the limiting triangle is equilateral.

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. IV.

SOLUTIONS of problems in No. 6, Vol. IV, have been rec'd as follows:

From Marcus Baker, 182 and 183; Prof. W. P. Casey, 182 and 186; Prof. A. B. Evans, 183; Geo. Eastwood, 183 and 185; Prof. J. M. Greenwood, 182, 183, 184 and 185; H. Heaton, 182, 183, 184, 185 and 186; W. E. Heal, 182 and 185; Prof. E. W. Hyde, 182, 183 and 186; Prof. W. W. Johnson, 186; Prof. J. H. Kershner, 182, 183 and 185; Christine Ladd, 186; Prof. D. J Mc. Adam, 182 and 185; Dr. A. B. Nelson, 183 and 185; Prof. A. T. Smith, 182; S. W. Salmon, 182, 183, 185, 186; Prof. Scheffer, 182, 183, 184, 185, 186; E. B. Seitz, 182—185, (for 186 see Chauv. Geom.)

182. "If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, show that $m^2 - n^2 = 4\sqrt{(mn)}$."

SOLUTION BY S. W. SALMON, MT. OLIVE, NEW JERSEY.

Taking the sum, difference and product of the equations we have

$$2\tan\theta = m+n \ldots (a), \qquad 2\sin\theta = m-n \ldots (b),$$

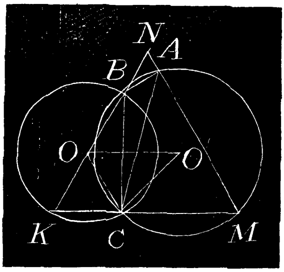
and $$mn = \tan^2\theta - \sin^2\theta = (\sin^2\theta - \cos^2\theta\sin^2\theta) \div \cos^2\theta = \sin^4\theta \div \cos^2\theta$$
$$= \tan^2\theta\sin^2\theta. \qquad (c)$$

Multiplying (a) by (b), we have $4\tan\theta\sin\theta = m^2 - n^2$; whence from (c), $m^2 - n^2 = 4\sqrt{(mn)}$.

183. "Show that the altitude of the greatest equilateral triangle that can be circumscribed about a given triangle, is $\sqrt{[a^2+b^2-2ab\cos(\frac{1}{3}\pi + C)]}$."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY.

Lht ABC be the given triangle and upon a and b describe arcs containing 60° each, O and O' being the centers of these arcs. Through C draw KM parallel to OO'. Draw MA and KB and prolong them to their intersecrion in N; then is KMN the maximum equilateral triangle that can be circumscribed about ABC. This is evident since KM, parallel to OO', is the longest line that can be drawn through C and terminated by the arcs O and O' respectively.



Now the altitude of the equilateral triangle KMN is equal to $\frac{1}{2}KM\sqrt{3} = OO'\sqrt{3}$. Considering the triangle $OO'C$ we see that $OC = a\sqrt{\frac{1}{3}}$, $O'C = b\sqrt{\frac{1}{3}}$ and $OCO' = \frac{\pi}{3} + C$; therefore we have

$$\overline{OO'}^2 = \tfrac{1}{3}(a^2+b^2-2ab\cos(\tfrac{\pi}{3}+C) \text{ or}$$
$$\overline{OO'}\sqrt{3} = \sqrt{[a^2+b^2-2ab\cos(\tfrac{\pi}{3}+C)]}.$$

184. "In cutting the maximum rectanguar parallelopipedon from a frustrum of a cone, five pieces are cut off. Find the volume of each of these pieces."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

The maximum parallelopipedon must have square bases, and the vertices of the upper base must be in the convex surface of the frustrum.

Let R, r = the radii of the bases of the frustrum, h = its altitude, h_1 = the altitude of the whole cone, v = the volume of the frustrum, x = the altitude of the parallelopipedon, $2y$ = the diagonal of its base, v_1 = its volume. Then $\quad x : R-y :: h : R-r$, or $h(R-y)-(R-r)x = 0 \quad (1)$

and $\quad v_1 = 2xy^2 =$ a maximum. $\quad (2)$

Differentiating (1) and (2), we have $hdy+(R-r)dx = 0$ and $2xdy+ydx = 0$, whence $\quad hy-2(R-r)x = 0. \quad (3)$

From (1) and (3) we find

$$x = \frac{hR}{3(R-r)},\ y = \tfrac{2}{3}R, \text{ and } v_1 = \frac{8hR^3}{27(R-r)}.$$

To get this parallelopipedon we may first cut from the top of the frustrum a piece whose volume is

$$v_2 = \frac{\pi h(8R^3-27r^3)}{81(R-r)}.$$

Two pieces from the opposite sides of the frustrum may then be cut off. To find the volume of one of these pieces, let it be intersected by a plane parallel to the base, at the distance z from the vertex of the cone; the area of the section will be

$$u = \frac{R^2z^2}{h_1^2}\cos^{-1}\left(\frac{h_1\sqrt{2}}{3z}\right)-\frac{R^2}{9h_1}\sqrt{\left(18z^2-4h_1^2\right)}.$$

Hence the volume is

$$v_3 = \int_{\frac{2}{3}h_1}^{h_1} udz = \frac{hR^3}{81(R-r)}\left[27\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)-2\pi+8-6\sqrt{14}\right.$$
$$\left.+2\sqrt{2}.\log\left(\frac{3+\sqrt{7}}{2+\sqrt{2}}\right)\right].$$

The volume of each of the remaining two pieces is

$$v_4 = \tfrac{1}{2}(v-v_1-v_2-2v_3).$$

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185. "Let the equation $ay^2+bxy+cx^2+dy+ex+f=0$ represent a parabola referred to rectangular axes. Prove that the latus rectum $= (bd-2ae)\div[(a+c)\sqrt{(4a^2+b^2)}]$."

SOLUTION BY PROF. J. M. GREENWOOD, KANSAS CITY, MO.

The general equation of the second degree is a parabola when the first three terms form a perfect square, and may be written thus:

$$a^2x^2+2abxy+b^2y^2+2gx+2fy+c = 0. \quad (1)$$

Comparing the given equation with (1), we have $c = a^2$, $a = b^2$, $g = \frac{1}{2}e$, $f = \frac{1}{2}d$ and $b = 2\sqrt{(ac)}$.

Salmon's Conics, page 186, fifth edition, the expression for the latus rectum is

$$p = \frac{2(fa - gb)}{(a^2 + b^2)^{\frac{3}{2}}}.$$

Writing $c = a^2$, $a = b^2$, $\frac{1}{2}e = g$, $\frac{1}{2}d = f$, and $b^2 = 4ac$, in the formula and reducing we get

$$p = (bd - 2ae) \div [(a + c)\sqrt{(4a^2 + b^2)}].$$

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186. "A_1, A_2; B_1, B_2 and C_1, C_2 are three pairs of points in a plane such that the three lines A_1A_2, B_1B_2, C_1C_2 meet in a common point O. Let B_1C_1 and B_2C_2 meet in a, and determine in like manner the points b and c. Prove that a, b and c are in a straight line.

Note. We have thus 10 points situated 3 by 3 on 10 lines, each point being the intersection of 3 lines, so that for each point there are three pairs of points colinear with it, and three points not joined to it in the figure but colinear with each other. Devise a notation which shall express the mutual relation of these poins and lines."

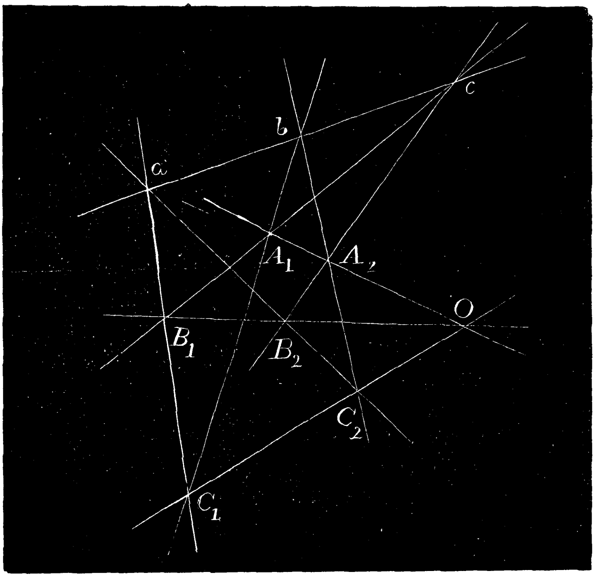
SOLUTION BY PROF. JOHNSON.

First suppose the three straight lines OA_1, OB_1, OC_1 *not* to be in the same plane; then, since the point a is on the line B_1C_1, it is in the plane $A_1B_1C_1$, and since it is on the line B_2C_2, it is in the plane $A_2B_2C_2$. In like manner b and c are at once in the plane $A_1B_1C_1$, and in the plane $A_2B_2C_2$; hence a, b and c are in the intersestion of these two planes, that is in a straight line. But, when the three lines OA_1, OB_1 and OC_1 are in the same plane, the figure may be regarded as the projection of a similarly formed figure in space, and the points a, b and c will be the projections of three points in a straight line, and therefore will themselves be in a straight line.

The mntual relations of the 10 points and the 10 lines of the figure are expressed by the following notation. Take any five symbols as a, b, c, d, e; then 10 combinations, two by two, may be used to denote the 10 points, and their 10 combinations, three by three, to express the 10 lines; in such a manner that the three points on $(a\ b\ c)$ are $(a\ b)$ $(a\ c)$ and $(b\ c)$, and the lines which pass through $(a\ b)$ are $(a\ b\ c)$ $(a\ b\ d)$ and $(a\ b\ e)$. Thus a line passes through a point, if two of their symbols agree; two points are joined by a line in the figure, if they agree in one of their symbols; and the three points which are not joined in the figure to the point denoted by a given pair of the symbols [as $(a\ b)$], are situated on the line denoted by the combination of the other three symbols. [as $(c\ d\ e)$.]

SOLUTION BY CHRISTINE LADD.

$A_1B_1C_1$ and $A_2B_2C_2$ are triangles having a centre of homology, O; hence, as is well known, they have an axis of homology $a\ b\ c$.



The notation of anharmonic ratios is the simplest that has hitherto been devised for expressing the relations of the points and lines of a geometrical net, whether plane or in space. According to that notation, the anharmoni ccoordinates of any point P, referred to a given triangle ABC, and a given point, O, are said to be x, y, z, when the following values obtain for anharmonics of pencils:-$(A.BOCP)=y\div z$, $(B.COAP)=z\div x$, $(C.AOBP=x\div y$. The point itself is then denoted by the symbol

$$P=(x,y,z).$$

If the triangle $A_1B_1C_1$ be taken as triangle of reference, we have the following symbols for points:—

$$\begin{array}{lll} A_1=(1,0,0), & B_1=(0,1,0), & C_1=(0,0,1); \\ A_2=(2,1,1), & B_2=(1,2,1), & C_2=(1,1,2); \\ a\ =(0,1,-1), & b\ =(-1,0,1), & c\ =(1,-1,0); \\ & O\ =(1,1,1). & \end{array}$$

Any three points are colinear when the determinant of their co-ordinates vanishes.

If the anharmonic co-ordinates of a line are l, m, n, the line itself is symbolically denoted by

$$L=[l,m,n].$$

For the lines connected with the figure, we have the following symbols:-

$$B_1C_1 = [1, 0, 0\ ],\quad C_1A_1 = [0, 1, 0\ ],\quad A_1B_1 = [0, 0, 1\ ];$$
$$O\,A_1 = [0, 1, -1],\quad O\,B_1 = [-1, 0, 1],\quad O\ C_1 = [1, -1, 0];$$
$$B_2C_2 = [-3, 1, 1],\quad C_2A_2 = [1, -3, 1],\quad A_2B_2 = [1, 1, -3];$$
$$a\,b\,c = [1, 1, 1\ ];$$

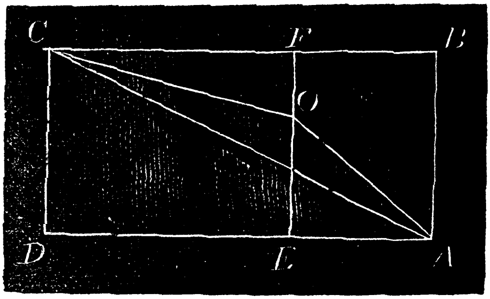
and any three are concurrent when the determinant of their coordinates vanishes. A point, P, is on a line, L, if $lx+my+nz$ vanishes identically.

This notation is especially adapted to reciprocations. In order to pass from a point to a line, it is only necessary to exchange the symbols ( ) and [ ].

[Professors Scheffer, Johnson and A. T. Smith, each answered the Query published at p. 176, Vol. IV, but want of room compels us to defer the publication of the answer to No. 2.]

PROBLEMS.

187. BY ISAAC H. TURRELL, CINCINNATI, OHIO.—Six circles may be described each of which shall touch four of the others. Prove that the lines joining the centers of the non-touching pairs, are concurrent.



188. BY HENRY C. ALLEN, NEW YORK CITY.—Two given rectangular planes, $ABFE$ and $EFCD$ lie contiguous, as in the figure. The plane BE is hard ground whereon one can travel at the rate $2m$; the other is soft ground whereon the rate of travel is m. Required the point O of crossing EF so that the time of journey from A to C shall be the quickest possible.

189. BY PROF. J. H. KERSHNER, MERCERSBURG, PA.—Find by Algebra the series whose sum is 127, when the sum of the squares of the seven terms is 5461.

190. BY PROF. ORSON PRATT, SEN.—Transform the equation,

$$x^9+9x^8+36x^7+85x^6+132x^5+141x^4+105x^3+54x^2+18x+9 = 0,$$

into an equation of the third degree.